# Establishing non-Abelian topological order in Gutzwiller projected Chern insulators via Entanglement Entropy and Modular $\mathcal{S}$-matrix

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We use entanglement entropy signatures to establish non-Abelian topological order in a new class of ground states, the projected Chern-insulator wave functions. The simplest instance is obtained by Gutzwiller projecting a filled band with Chern number $C = 2$ which may also be viewed as the square of the band insulator Slater determinant. We demonstrate that this wave function is captured by the $SU(2)_2$ Chern Simons theory coupled to fermions. In addition to the expected torus degeneracy and topological entanglement entropy, we also show that the modular $\mathcal{S}$-matrix, extracted from entanglement entropy calculations, provides direct access to the peculiar non-Abelian braiding statistics of Majorana fermions in this state. We also provide microscopic evidence for the field theoretic generalization, that the N$^{\text{th}}$ power of a Chern number $C$ Slater determinant realizes the topological order of the $SU(N)_C$ Chern Simons theory coupled to fermions, by studying the $SU(2)_3$ and the $SU(3)_2$ wave functions. An advantage of projected Chern insulator wave functions over lowest Landau level wave functions for the same phases is the relative ease with which physical properties, such as entanglement entropy, can be numerically calculated using Monte Carlo techniques.

PACS numbers:



It is known that quasiparticles may go beyond the conventional bosonic and fermionic statistics in two-dimensional many-body systems. Such systems represent a new class of phases not characterized by conventional order but by topological order[1]. One famous example is the Abelian quantum hall liquid for a fractionally filled Landau level in an interacting two-dimensional electron liquid[2] and the corresponding Laughlin wave function[3], where the quasiparticles exhibit fractional statistics. The low-energy effective theory is the $U(1)$ Chern Simons (CS) theory and is relatively well understood[4].

There has been much recent interest in an even more exotic generalization of statistics, non-Abelian statistics, displayed by phases with non-Abelian topological order. There is an increasing interest partially brought on by the recent preliminary evidence of Majorana fermions in superconductor-semiconductor junctions[5] and their potentials for topological quantum memory and computation realizations[6–8]. Other examples include the $\nu = 5/2$ fractional quantum hall effect[9] and the Moore-Read Pfaffian state[10]. Although a general theory and classification of such states is still absent, there is a wide class effectively characterized by non-Abelian higher-level CS theories[11]: the low-energy theory for filling $\nu = k/N$ over $k$ Landau levels is the $SU(N)_k$ CS theory. whose low energy effective theory is the $SU(2)_2$ CS theory. The elementary excitations are Ising anyons, whose braiding transforms the ground state, instead of just incur additional phase factors as in the Abelian cases[12]. Its entanglement spectrum has also been established as a powerful tool of identification[13]. For a universal topological quantum computation, the complexity of $SU(2)_3$ topological order with Fibonacci anyons (or above) is necessary[8], which may describe the fractional quantum hall plateau at $\nu = 12/5$. Examples of such phases are given by the Read-Rezayi states[14] and generalizations to lattice spin liquid states[15,16], however, their physical properties are relatively difficult to evaluate.

For fractional quantum Hall liquids, extreme experimental conditions such as clean samples and large magnetic fields are necessary to increase interaction and suppress Landau level mixing. Yet it is later realized that the a Landau level structure is not necessary[17], as the quantum Hall effects is also present in Chern insulators - lattice band models without an external magnetic field but carry nontrivial reciprocal space Berry curvatures. It also offers a new arena for topological orders, and there have been various evidences for Abelian[18–22] and non-Abelian[23–26] fractional Chern insulators.

Recall, the Laughlin wave functions[3] $\psi \sim \prod (z_i - z_j)^m e^{-|z_k|^2/4}$, can be considered as the $m^{\text{th}}$ power of a integer Quantum Hall state of a particle with reduced charge, that fills the lowest Landau level. Previously, we confirmed that the lattice analog of this statement: the $m^{\text{th}}$ power of a Chern band wave function with unit Chern number $\psi \sim \psi^m_{C=1}$ has the topological order of a Laughlin state of order $m$[27,28]. In this paper, we focus on the cases when the Chern number $C > 1$, which is unique to the lattice models and has no simple Landau level counterpart. Consistent with the field theory study in Ref. 29 and parton construction scheme proposed in Ref. 30, we suggest that the square (power $N = 2$) of $C = 2$ Chern band wave functions $\psi \sim \psi^2_{C=2}$ are captured by the $SU(2)_2$ CS theory coupled to fermions and have the same quasiparticle braiding statistics as the Moore Read Pfaffian state[40] $\psi \sim \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod (z_i - z_j)\, e^{-|z_k|^2/4}$. We verify such ground states are only three-fold degenerate. Especially, the wave-function diagnostic algorithm in Ref. 28 can be generalized to non-Abelian cases and particularly useful for many-body system where entanglement spectrum

is not available. With variational ansatz, physical measurable of these states are much simpler to calculate, thus we are able to extract the modular $\mathcal{S}$-matrix easily and determine the quantum dimension and quasiparticle statistics through topological entanglement entropy (TEE)[27,28,31,32], and prove the existence of non-Abelian quasiparticles. To our knowledge, this is the first direct numerical measurement of the modular $\mathcal{S}$-matrix and identification of a non-Abelian topological order wave function. We also extend our studies of ground-state degeneracy and entanglement to the generalized cases of $C = 2, N = 3$ ($\psi \sim \psi^3_{C=2}$) and $C = 3, N = 2$ ($\psi \sim \psi^2_{C=3}$). These results imply the effective theory for the $N^{\text{th}}$ power of a $C = k$ Chern insulator's band $\psi \sim \psi^N_{C=k}$ is the $SU(N)_k$ CS theory coupled to fermions, allowing non-Abelian statistics when $N > 1$ and $k > 1$. Besides, one may have access to the entire ground-state manifold with choices of boundary conditions of the parent Chern insulator.

In the construction of a chiral spin liquid wave function within the slave particle formalism[27,33–35], the spin operator may be expressed as $s^\dagger = f^\dagger_\uparrow f_\downarrow$, where $f_\uparrow$ and $f_\downarrow$ are annihilation operators for spin up and spin down fermions. If the $f$ fermions fill up a band with Chern number $C = 1$ and is restricted to one fermion per site Hilbert space by Gutzwiller projection, it will be a candidate wave function for chiral spin liquid. Here we extend the situations to Chern number $C > 1$ to look for occurrence of non-Abelian statistics.

*Chern Number $C = 2$ Model:* To construct a two-band model with Chern number $C = \pm 2$, consider a tight-binding model on a two-dimensional square lattice with two orbitals on each lattice site labeled by $I = 1, 2$:

$$\begin{aligned} H &= \sum_{<ij>,I} (-1)^{I+1} c^\dagger_{jI} c_{iI} + \sum_{<ij>} \left( e^{i2\theta_{ij}} c^\dagger_{j2} c_{i1} + h.c. \right) \\ &+ \delta \sum_{<<ik>>} \left( e^{i2\theta_{ik}} c^\dagger_{k2} c_{i1} + h.c. \right) \end{aligned} \tag{1}$$

where $\theta_{ij}$ is the azimuthal angle for the vector connecting $i$ and $j$. We rewrite the Hamiltonian in momentum space:

$$\begin{aligned} H_k &= \vec{g}(k) \cdot \vec{\sigma} \\ g_1 &= 2(\cos k_x - \cos k_y) \\ g_2 &= 2\delta \left[\cos(k_x + k_y) - \cos(k_x - k_y)\right] \\ g_3 &= 2(\cos k_x + \cos k_y) \end{aligned} \tag{2}$$

where $\sigma_i$ are Pauli matrices denoting the two orbitals. The model has a finite gap between the two bands with Chern number $C = \pm 2$ respectively. We have also verified $C = \pm 2$ numerically by counting the number of chiral modes on the physical edges as well as within the entanglement spectrum. Hereafter, we assume $\delta = 1/\sqrt{2}$ for a maximum gap to suppress the finite size effect.

*Torus Ground States of Gutzwiller Projected States and $SU(2)_2$ CS Theory:* Similar to the chiral spin liquid case[27], Eqn. 1 satisfies the Marshall sign rule, so the spin wave function may simply be express as the square of the fermionic wave function $\chi^2(z_1, z_2 \cdots)$ upto an unimportant phase, where $\chi(z_1, z_2 \cdots)$ is the ground-state wave function of lattice Chern insulator in Eqn. 1 and $z = (r, I)$ contains both the position and orbital index. Its properties are the major focus of this paper. Note that $\chi^2(z_1, z_2 \cdots)$ is $\pi/2$ rotational symmetric even though $\chi(z_1, z_2 \cdots)$ is not; although the $\pi/2$ rotation symmetry is not essential to the topological properties, it will be convenient for determining them.

Physical quantities related with the wave functions may be calculated with variational Monte Carlo method[36]. We study the wave functions on a torus with periodic or antiperiodic boundary conditions along the $\hat{x}$ and $\hat{y}$ directions in Eqn. 1. Note that the squared wave function $\chi^2(z_1, z_2 \cdots)$ is insensitive to such boundary conditions, hence all four such wave functions are equally good candidate ground states, denoted as $|\Phi_x \Phi_y >$ where $\Phi_{x,y} = 0, \pi$. To understand their linear dependence and the ground-state manifold, we calculate the overlaps between them with variational Monte Carlo method on a $12 \times 12$ system:

$$\begin{aligned} \langle 00|\pi\pi\rangle &= \alpha \\ \langle 0\pi|\pi 0\rangle &= \alpha' \\ \langle 0\pi|00\rangle &= \langle \pi 0|00\rangle = \beta \\ \langle 0\pi|\pi\pi\rangle &= \langle \pi 0|\pi\pi\rangle = \beta' \end{aligned} \tag{3}$$

Numerically we find to very high accuracy that $\alpha = \alpha' = -0.086$ and $\beta = \beta' = 0.457$. We may construct a "generalized" projection operator:

$$\begin{aligned} P &= \sum |\Phi_x \Phi_y\rangle \langle \Phi_x \Phi_y| \, |\Phi'_x \Phi'_y\rangle \langle \Phi'_x \Phi'_y| \\ &= \begin{pmatrix} |\pi 0\rangle \\ |0\pi\rangle \\ |\pi\pi\rangle \\ |00\rangle \end{pmatrix}^T \begin{pmatrix} 1 & \alpha & \beta & \beta \\ \alpha & 1 & \beta & \beta \\ \beta & \beta & 1 & \alpha \\ \beta & \beta & \alpha & 1 \end{pmatrix} \begin{pmatrix} \langle \pi 0| \\ \langle 0\pi| \\ \langle \pi\pi| \\ \langle 00| \end{pmatrix} \end{aligned} \tag{4}$$

which projects to the ground state manifold. Due to the non-orthogonality between the basis states, the eigenvalues of $P$ contains one 0, so the corresponding eigenstate is projected out:

$$P\left[|\pi\pi> - |0\pi> - |\pi 0> + |00>\right] = 0 \tag{5}$$

where we have used $2\beta = 1 + \alpha$ (true to high numerical accuracy). Thus the ground states are only three-fold degenerate on a torus:

$$\begin{aligned} |F_x = 1, F_y = 1> &\simeq (|00> + |0\pi> + |\pi 0> + |\pi\pi>) \\ |F_x = 1, F_y = -1> &\simeq (|00> - |0\pi> + |\pi 0> - |\pi\pi>) \\ |F_x = -1, F_y = 1> &\simeq (|00> + |0\pi> - |\pi 0> - |\pi\pi>) \end{aligned} \tag{6}$$

upto phase and normalization, where we have used the basis of flux threading operators $F_x$ and $F_y$ to label these

states. The ground-state degeneracy and linear dependence is consistent with the $SU(2)_2$ CS theory, as shown in Ref. 41. In addition to these simple choices of boundary conditions, there are other possible constructions for ground states, where the wave function is a product of two fermionic wave functions with different boundary conditions. These are shown to be linearly dependent on the wave functions above[41].

*Topological Entanglement Entropy:* To obtain further information on the wave functions, we use entanglement as a tool and extract their TEE on a $6 \times 6$ system with a method by Kitaev and Preskill[31], which effectively cancels out the boundary and corner contributions and exposes the topological term when the typical length scale is much longer than the correlation length. Although the smallest length scale is only 2 lattice spacings for the system size we study, it is still longer than the correlation length $\xi \sim 0.5$ lattice spacing. In addition, the corresponding wave-function overlaps suggest that the residue of Eqn. 5 is just $\sim 1.3\%$, thus the three-fold degeneracy in Eqn. 6 is still a good approximation. These facts suggest that the system size is large enough to suppress finite size effect and maintain topological properties. Throughout we focus on the Renyi entropy $S_2$ for algorithmic simplifications[27,28,37].

We find that the TEE of $|\Phi_x \Phi_y>$ for a topologically trivial disc-shape cut is $\gamma = 0.85 \pm 0.08$, in fairly good consistency with the theoretical value $\lambda_{SU(2)_2} = \log 2 \sim 0.693$ for the $SU(2)_2$ CS theory. Note that for an Abelian topological order with $D^2 = 3$ degenerate ground states on a torus, the expected TEE $\gamma = \log D \sim 0.549$ is further away from the calculated value and unlikely to describe the wave functions. It implies non-Abelian statistics.

*Modular S-Matrix from Entanglement Entropy:* Next we extract the quasiparticle braiding information using entanglement. Following the algorithm in Ref. 28, we (i) calculate the TEE $\gamma'$ for partitioning the torus into two cylinders along the $\hat{y}$ direction for linear combinations of wave functions, see Fig. 1 inset, (ii) search for the states with minimum entanglement entropy (maximum TEE $\gamma'$) and identify them as the quasiparticle Wilson loop states, and (iii) study the $\pi/2$ rotation transformation on them. The resulting TEE $2\gamma - \gamma'$ for $|\Phi_1> = \cos\theta|0\pi> + \sin\theta|\pi 0>$, $|\Phi_2> = \sin\theta|00> - \cos\theta|\pi 0>$, and:

$$\begin{aligned}|\Phi_3> &= (\sin\theta + 0.7915\cos\theta)|00> \\ &- (\sin\theta + 0.4697\cos\theta)|\pi 0> \\ &- 1.2623\cos\theta|0\pi> \end{aligned} \tag{7}$$

for selected values of $\theta$ is shown in Fig. 1, Fig. 2a and Fig. 2b, respectively.

Then we have obtained approximately the three orthogonal quasiparticle Wilson loop states corresponding

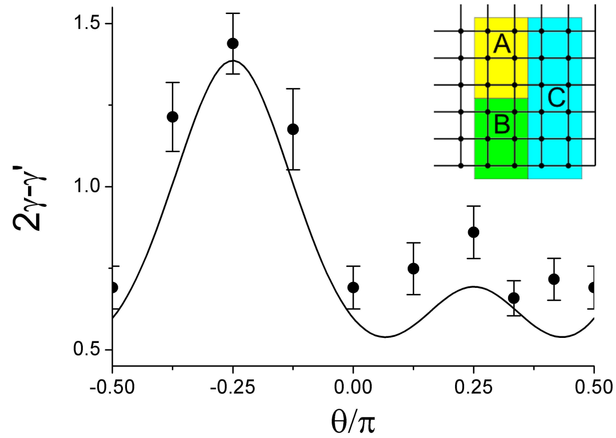


FIG. 1: (Inset) Kitaev Preskill scheme for extracting TEE by partition the torus into subsystems A, B and C and constructing $\Delta = -S_{ABC} + S_{AB} + S_{BC} + S_{AC} - S_A - S_B - S_C = -2S_A + 2S_{AB} - S_{ABC}$[31]. Note here regions $C$ and $AB$ encircle the torus leading to TEE of $\gamma'$, which is ground state dependent, unlike the TEE for region A (or B) which is always $\gamma$[28,37]. The resulting $\Delta = 2\gamma - \gamma'$ is plotted for the linear combinations of wave functions $|\Phi_1>$. The solid curve is expected value for the $SU(2)_2$ CS theory.

to the minima of $2\gamma - \gamma'$[28]:

$$\begin{aligned}|\Xi_1> &= -|\pi 0> - |00> \\ |\Xi_2> &= -|\pi 0> + |00> \\ |\Xi_3> &= 0.7915|00> - 0.4697|\pi 0> - 1.2623|0\pi> \end{aligned} \tag{8}$$

For a $\pi/2$ rotation symmetric system, the modular $\mathcal{S}$-matrix's action in the $|\Xi_j>$ basis is a $\pi/2$ rotation. Note the transformations $|\pi 0> \leftrightarrow |0\pi>$ and $|00> \leftrightarrow |00>$ under $\pi/2$ rotation, we may transform the basis to Eqn. 8 with the help of Eqn. 3 to properly treat orthogonality:

$$\mathcal{S} = \begin{pmatrix} 0.627 & 0.610 & 0.484 \\ 0.610 & 0.000 & -0.792 \\ 0.484 & -0.792 & 0.372 \end{pmatrix} \tag{9}$$

The rough estimation of the modular $\mathcal{S}$-matrix in Eqn. 9 already suggests important qualitative results for quasiparticle braiding. Especially, the phase for the $ij$'th entry of the $\mathcal{S}$-matrix is the phase factor during the process of creating the i'th and j'th quasiparticle, braiding the quasiparticles before annihilating them with each anti-quasiparticles and returning to the ground states. While the quasiparticles corresponding to the first and third rows and columns obey Abelian bosonic or fermionic statistics for a trivial phase upon braiding, the zero value entry for the second quasiparticle is a signature to its non-Abelian self statistics. Indeed, for the Majorana fermion in the $SU(2)_2$ CS theory, one fermion composes a pair of Majorana fermions $c = \gamma_1 + i\gamma_2$, however, when one

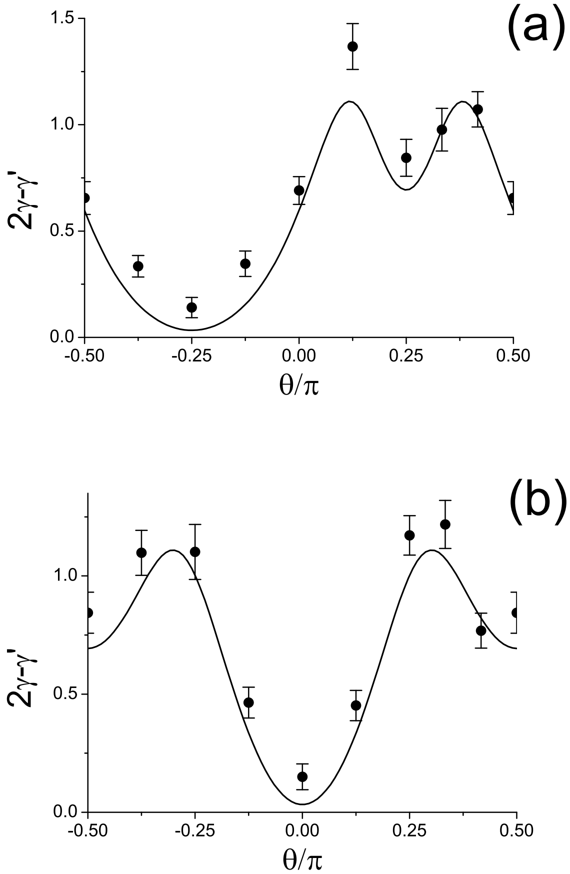


FIG. 2: TEE $\Delta = 2\gamma - \gamma'$ for linear combinations of wave functions: (a) $|\Phi_2 >$, and (b) $|\Phi_3 >$. The solid curves are expected values for the $SU(2)_2$ CS theory. Note, the presence of only two minima with $\Delta \simeq 0$ indicates that two of the three quasiparticles are Abelian, while the third is non-Abelian.

of the Majorana fermions braids with another Majorana fermion and picks an additional $\pi$ phase, it changes the fermion annihilation operator to creation operator and vice versa, and fails to return to the ground state afterwards, thus the corresponding zero entry in the modular $\mathcal{S}$-matrix.

As a comparison, the ideal $\mathcal{S}$-matrix for the $SU(2)_2$ CS theory is:

$$\mathcal{S} = \begin{pmatrix} 0.5 & 0.707 & 0.5 \\ 0.707 & 0 & -0.707 \\ 0.5 & -0.707 & 0.5 \end{pmatrix} \tag{10}$$

roughly consistent with Eqn. 9. Note that although the $|\Xi_j >$ states in Eqn. 8 are estimates, the qualitative features from the corresponding $\mathcal{S}$-matrix are more robust.

For a more detailed comparison, we note the $SU(2)_2$ CS theory predicts the following connection between the ground state wave functions of Eqn.6 and the anyons of this phase:[12]:

$$\begin{aligned} |F_x = 1, F_y = 1 > &= (|1_y > + |\psi_y >)/\sqrt{2} \\ |F_x = 1, F_y = -1 > &= (|1_y > - |\psi_y >)/\sqrt{2} \\ |F_x = -1, F_y = 1 > &= |\sigma_y > \end{aligned} \tag{11}$$

where $|1_y >$, $|\psi_y >$ and $|\sigma_y >$ are the Wilson loop states along the $\hat{y}$ direction for the identity, fermionic and non-Abelian Ising anyonic quasiparticles, respectively. Then for an arbitrary ground state:

$$2\gamma - \gamma' = -\log\left(\sum_j p_j^2/d_j^2\right) \tag{12}$$

where $d_j$ is the individual quantum dimension and $p_j$ is the statistical weight in the wave function for the $j^{\text{th}}$ quasiparticle[38]. Combining Eqn. 3, Eqn. 6 and Eqn. 11 with relative phase between wave functions therein, one may expand the $|\Phi_x\Phi_y >$ states in the $|1_y >$, $|\psi_y >$, $|\sigma_y >$ basis. The corresponding values of $2\gamma - \gamma'(\theta)$ given by Eqn. 12 and $d_1 = d_\psi = 1, d_\sigma = \sqrt{2}$ are shown in Fig. 1 and Fig.2 as the solid curves, and fit well with the numerical results. These individual quantum dimensions $d_j$ are also implicit in the values of the minima as $2\gamma - \gamma_j' = \log(d_j^2)$, which directly follows from Eqn. 12. Especially, $d_\sigma = \sqrt{2}$ suggests the $\sigma$ quasiparticle obeys non-Abelian statistics.

*Other non-Abelian States* *(i)* $SU(3)_2$: Such construction of wave functions may be generalized to more complicated non-Abelian cases. As another example, for the wave functions $\chi^3(z_1, z_2 \cdots)$, we can construct nine candidate ground-state wave functions $|\Phi_x\Phi_y >$ with simple boundary conditions $\Phi_{x,y} = 0, \pm 2\pi/3$. Repeating the procedure for Eqn.3 and Eqn. 4, we obtain its "generalized" projection operator $P'$ on a $12 \times 12$ torus, see Ref. 41. Indeed, $P'$ only has six non-zero eigenvalues, therefore the ground states are six-fold degenerate. The six corresponding eigenstates' $\pi/2$ rotation eigenvalues are $\pm 1$, $\pm 1$ and $\pm i$. These results are consistent with the $SU(3)_2$ CS theory. In addition, the TEE for a disc-shaped cut on a $6 \times 6$ torus is $\gamma \simeq 1.264 \pm 0.073$, consistent with the theoretical value of $D = \sqrt{3(5 + \sqrt{5})/2}$ and $\gamma_{SU(3)_2} = \log D \simeq 1.19$. While we consider the above evidences sufficient, we leave further verifications as TEE ground-state dependence and linear dependence of more complex boundary constructions to future works.

*(ii)* $SU(2)_3$ *in close connection to the Read-Rezayi state:* In the same spirit, to construct wave functions for the Fibonacci anyons characterized by the $SU(2)_3$ CS theory, we may construct the following tight-binding model on a triangular lattice with two orbitals on each site labeled by $I = 1, 2$ in analogy to Eqn. 1:

$$\begin{aligned} H &= \sum_{<ij>, I} (-1)^{I+1} c_{jI}^\dagger c_{iI} + \sum_{<ij>} \left(e^{i3\theta_{ij}} c_{j2}^\dagger c_{i1} + h.c.\right) \\ &+ \delta \sum_{<<ik>>} \left(e^{i3\theta_{ik}} c_{k2}^\dagger c_{i1} + h.c.\right) \end{aligned} \tag{13}$$

where $\theta_{ij}$ is the azimuthal angle for the vector connecting $i$ and $j$. The model is a two-band Chern insulator with Chern number $C = \pm 3$ for $\delta \neq 0$. Similar construction may have potentials for the construction of even higher Chern number bands, and a systematic scheme to produce arbitrary Chern number bands has been studied in Ref. 26. We choose nine different boundary conditions in Eqn. 13 for the construction of $\chi^2(z_1, z_2 \cdots)$ on a $12 \times 12$ torus. Our results show that only four of the nine eigenvalues of the corresponding "generalized" projection operator have significantly finite values, so the ground state is only four-fold degeneracy, consistent with the $SU(2)_3$ CS theory.

These examples indicate that the $N^{\text{th}}$ power of a Chern insulator with Chern number $C = k$ is characterized by the $SU(N)_k$ CS theory (coupled to fermions), thus providing a novel route to construct wave functions for non-Abelian topological order, and the power of entanglement in the diagnosis and study of these wave functions.

In conclusion we have introduced a class of lattice wave functions for non-Abelian topological phases that (i) are readily generalized to capture $SU(N)_k$ topological order and (ii) easily generate all degenerate ground states and (iii) whose properties can be compute with using montecarlo techniques. Further the presence of such natural lattice wave functions holds promise that such states may be realized in the context of fractional Chern insulators.

**Acknowledgements:**

We thank Yuan-ming Lu, Tarun Grover, Ying Ran and Maissam Barkeshli for helpful discussions. This work is supported by NSF DMR-1206728.

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[1] X.-G. Wen, Int. J. Mod. Phys. B **4**, 239 (1990).
[2] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).
[3] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[4] X.-G. Wen, Advances in Physics **44**, 405 (1995).
[5] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven, Science (2012).
[6] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, Journal of Mathematical Physics **43**, 4452 (2002).
[7] A. Kitaev, Annals of Physics **303**, 2 (2003), ISSN 0003-4916.
[8] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. **80**, 1083 (2008).
[9] R. Willett, J. P. Eisenstein, H. L. Störmer, D. C. Tsui, A. C. Gossard, and J. H. English, Phys. Rev. Lett. **59**, 1776 (1987).
[10] G. Moore and N. Read, Nuclear Physics B **360**, 362 (1991), ISSN 0550-3213.
[11] X. G. Wen, Phys. Rev. Lett. **66**, 802 (1991).
[12] M. Oshikawa and T. Senthil, Phys. Rev. Lett. **96**, 060601 (2006).
[13] H. Li and F. D. M. Haldane, Phys. Rev. Lett. **101**, 010504 (2008).
[14] N. Read and E. Rezayi, Phys. Rev. B **59**, 8084 (1999).
[15] M. Greiter and R. Thomale, Phys. Rev. Lett. **102**, 207203 (2009).
[16] B. Scharfenberger, R. Thomale, and M. Greiter, Phys. Rev. B **84**, 140404 (2011).
[17] F. D. M. Haldane, Phys. Rev. Lett. **60**, 635 (1988).
[18] N. Regnault and B. A. Bernevig, Phys. Rev. X **1**, 021014 (2011).
[19] T. Neupert, L. Santos, C. Chamon, and C. Mudry, Phys. Rev. Lett. **106**, 236804 (2011).
[20] Y.-F. Wang, Z.-C. Gu, C.-D. Gong, and D. N. Sheng, Phys. Rev. Lett. **107**, 146803 (2011).
[21] D. Sheng, Z.-C. Gu, K. Sun, and L. Sheng, Nat Commun **2**, 389 (2011).
[22] J. W. F. Venderbos, S. Kourtis, J. van den Brink, and M. Daghofer, Phys. Rev. Lett. **108**, 126405 (2012).
[23] Y.-F. Wang, H. Yao, Z.-C. Gu, C.-D. Gong, and D. N. Sheng, Phys. Rev. Lett. **108**, 126805 (2012).
[24] B. A. Bernevig and N. Regnault, Phys. Rev. B **85**, 075128 (2012).
[25] A. Sterdyniak, C. Repellin, B. A. Bernevig, and N. Regnault, ArXiv e-prints: 1207.6385.
[26] S. Yang, Z.-C. Gu, K. Sun, and S. Das Sarma, ArXiv e-prints: 1205.5792.
[27] Y. Zhang, T. Grover, and A. Vishwanath, Phys. Rev. B **84**, 075128 (2011).
[28] Y. Zhang, T. Grover, A. Turner, M. Oshikawa, and A. Vishwanath, Phys. Rev. B **85**, 235151 (2012).
[29] X.-G. Wen, Phys. Rev. B **60**, 8827 (1999).
[30] Y.-M. Lu and Y. Ran, Phys. Rev. B **85**, 165134 (2012).
[31] A. Kitaev and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006).
[32] M. Levin and X.-G. Wen, Phys. Rev. Lett. **96**, 110405 (2006).
[33] V. Kalmeyer and R. B. Laughlin, Phys. Rev. Lett. **59**, 2095 (1987).
[34] V. Kalmeyer and R. B. Laughlin, Phys. Rev. B **39**, 11879 (1989).
[35] X. G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B **39**, 11413 (1989).
[36] C. Gros, Annals of Physics **189**, 53 (1989), ISSN 0003-4916.
[37] Y. Zhang, T. Grover, and A. Vishwanath, Phys. Rev. Lett. **107**, 067202 (2011).
[38] S. Dong, E. Fradkin, R. G. Leigh, and S. Nowling, Journal of High Energy Physics **2008**, 016 (2008).
[39] M. Barkeshli and X.-G. Wen, Phys. Rev. B **81**, 155302 (2010).
[40] Strictly speaking this wave function is not the same topological order as the Moore Read Pfaffian state captured by the pure $SU(2)_2$ CS theory; although it shares the same ground-state degeneracy and quasiparticle excitation braiding encoded in the modular $\mathcal{S}$-matrix with the Pfaffian state, it differs at the level of the quasiparticle spin encoded in the modular $T$-matrix due to the fermionic core for certain quasiparticles. Similar difference exists between the $SU(2)_k$ states we study and the Read-Rezayi states. See Ref. 39 for detailed discussion. We do not focus on such distinctions because these states behave identical for the properties we study.
[41] See supplementary material online.

### Appendix A: Parton boundary conditions for Chern Simons Theories

The $SU(N)_k$ CS theory is :

$$\mathcal{L}_{SU(N)_k} = \frac{k}{4\pi}\epsilon_{\mu\nu\lambda}Tr\left(\hat{a}_\mu\partial_\nu\hat{a}_\lambda + \frac{i}{3}\hat{a}_\mu\hat{a}_\nu\hat{a}_\lambda\right), \hat{a}_\mu \in SU(N)$$

In this appendix we first study the boundary conditions and linear dependence for various wave functions through parton construction for $SU(2)_2$ CS theory. After choosing gauge $a_0 = 0$ and $\hat{a} = a \cdot Diag(1, -1)$, we have:

$$\mathcal{L}_{SU(2)_2} = \frac{4}{4\pi}a_\mu\partial_\nu a_\lambda$$

denoting low energy degrees of freedom $u = a_1 L_1/2\pi$ and $v = a_2 L_2/2\pi$, both defined modulo 1, the effective action becomes:

$$\mathcal{L}_{eff} = 8\pi v\dot{u}$$

therefore the canonical commutation relation:

$$[u, v] = \frac{i}{8\pi}$$

so the conjugate momentum of $u$ is $8\pi v$, with $(u, v) \sim (-u, -v)$ due to global $SU(2)$ gauge transformations. Then if the wave function in the $v$ coordinate is:

$$\phi(v) = \sum c_n\delta(4v - n)$$

which is essentially the Wilson loop states in the $\hat{y}$ direction. Similarly, the wave function in the $u$ coordinate is:

$$\phi(u) = \sum c_n e^{i2\pi nu}$$

As a result, in the spirit of parton construction, we may write down the following wave functions:

$$\begin{aligned} |1_y> &= \left[\phi(0,0) + \phi(1/2,0) + \sqrt{2}\phi(1/4,0)\right]/2 \\ |\psi_y> &= \left[\phi(0,1/2) + \phi(1/2,1/2) - \sqrt{2}\phi(1/4,1/2)\right]/2 \\ |\sigma_y> &= \left[\phi(0,1/4) - \phi(1/2,1/4)\right]/\sqrt{2} \end{aligned}$$

and:

$$\begin{aligned} (|1_y> - |\psi_y>)/\sqrt{2} &= [\phi(1/4,0) - \phi(1/4,1/2)]/\sqrt{2} \\ (|1_y> + |\psi_y>)/\sqrt{2} &= [\phi(0,0) + \phi(1/2,0) \\ &+ \phi(0,1/2) + \phi(1/2,1/2)]/2 \end{aligned}$$

here eight wave functions with different parton boundary conditions are involved:

$$\begin{aligned} \phi(0,0) &= \chi_{0,0}^2 \\ \phi(0,1/2) &= \chi_{0,\pi}^2 \\ \phi(1/2,0) &= \chi_{\pi,0}^2 \\ \phi(0,0) &= \chi_{\pi,\pi}^2 \end{aligned}$$

which have the same boundary condition for both parton wave functions $\chi$, and:

$$\begin{aligned} \phi(0,1/4) &= \chi_{0,\pi/2}\chi_{0,-\pi/2} \\ \phi(1/2,1/4) &= \chi_{\pi,\pi/2}\chi_{\pi,-\pi/2} \\ \phi(1/4,0) &= \chi_{\pi/2,0}\chi_{-\pi/2,0} \\ \phi(1/4,1/2) &= \chi_{\pi/2,\pi}\chi_{-\pi/2,\pi} \end{aligned}$$

where the boundary conditions for two partons are different.

However, these wave functions are not completely linear independent. Comparing the rotation eigenstates and the Wilson loop states, we obtain the following linear dependence:

$$\begin{aligned} &[\phi(0,1/4) - \phi(1/4,0) + \phi(1/4,1/2) - \phi(1/2,1/4)]/2 \\ &= [\phi(0,1/2) - \phi(1/2,0)]/\sqrt{2} \\ &[\phi(0,1/4) + \phi(1/4,0) + \phi(1/4,1/2) + \phi(1/2,1/4)]/2 \\ &= [\phi(0,0) + \phi(1/2,0) + \phi(1/2,0) + \phi(1/2,1/2)]/2 \\ &[\phi(0,1/4) + \phi(1/4,0) - \phi(1/4,1/2) - \phi(1/2,1/4)]/2 \\ &= [\phi(0,0) - \phi(1/2,1/2)]/\sqrt{2} \\ &\phi(0,1/4) - \phi(1/4,0) - \phi(1/4,1/2) + \phi(1/2,1/4) = 0 \end{aligned}$$

So the latter four wave functions may be fully established from the former four. In addition:

$$\phi(0,0) - \phi(0,1/2) - \phi(1/2,0) + \phi(1/2,1/2) = 0$$

which we have numerically verified as Eqn. 5 in the main text of the paper. To verify the other four linear dependence relations, we obtain numerically the following overlaps on a $12 \times 12$ torus between the wave functions with relevant boundary conditions in Eqn. 1:

$$\begin{aligned} \langle 00|0\sigma\rangle &= \langle 00|\sigma 0\rangle \simeq \langle\pi\pi|\pi\sigma\rangle = \langle\pi\pi|\sigma\pi\rangle \\ &\simeq \langle\pi 0|\pi\sigma\rangle \simeq \langle 0\pi|0\sigma\rangle \simeq 0.8403 \\ \langle 00|\pi\sigma\rangle &= \langle 00|\sigma\pi\rangle \simeq \langle\pi\pi|0\sigma\rangle = \langle\pi\pi|\sigma 0\rangle \\ &\simeq \langle 0\pi|\pi\sigma\rangle \simeq \langle\pi 0|0\sigma\rangle \simeq 0.0726 \end{aligned}$$

where $|0\sigma\rangle$, $|\pi\sigma\rangle$, $|\sigma 0\rangle$ and $|\sigma\pi\rangle$ correspond to $\phi(0,1/4)$, $\phi(1/2,1/4)$, $\phi(0,1/2)$ and $\phi(1/4,1/2)$, respectively. Combining these results with Eqn. 3, one can easily reproduce the rest four relations.

Overall, there are five constraints so leave only three linearly independent wave functions, consistent with the three-fold ground-state degeneracy in a $SU(2)_2$ CS theory.

The above discussion may be generalized to parton constructions of other $SU(N)_k$ CS theories. It indicates certain relations of wave functions constructed from various boundary conditions, and implies a method to verify or identify the underlying topological theory of a set of parton wave functions just by calculating their overlaps.

As a second example, we calculate the wave-function overlap for $\chi^3(z_1, z_2\cdots)$ of Eqn. 1 on a $12\times 12$ system. In the basis of $(|00\rangle, \left|0\frac{2\pi}{3}\right\rangle, \left|0\frac{-2\pi}{3}\right\rangle, \left|\frac{2\pi}{3}0\right\rangle, \left|\frac{2\pi}{3}\frac{2\pi}{3}\right\rangle, \left|\frac{2\pi}{3}\frac{-2\pi}{3}\right\rangle, \left|\frac{-2\pi}{3}0\right\rangle, \left|\frac{-2\pi}{3}\frac{2\pi}{3}\right\rangle, \left|\frac{-2\pi}{3}\frac{-2\pi}{3}\right\rangle)$, the corresponding "generalized" projection operator is:

$$P' = \begin{pmatrix} 1 & a & a & a & b & b & a & b & b \\ a & 1 & a & bc^* & ac^* & bc^* & bc & ac & bc \\ a & a & 1 & bc & ac & bc & bc^* & bc^* & ac^* \\ a & bc & bc^* & 1 & ac & ac^* & a & bc & bc^* \\ b & ac & bc^* & ac^* & 1 & ac & bc & ac^* & b \\ b & bc & ac^* & ac & ac^* & 1 & bc^* & b & ac \\ a & bc^* & bc & a & bc^* & bc & 1 & ac^* & ac \\ b & ac^* & bc & bc^* & ac & b & ac & 1 & ac^* \\ b & bc^* & ac & bc & b & ac^* & ac^* & ac & 1 \end{pmatrix}$$

where $a \simeq 0.4213$, $b \simeq 0.0768 \simeq 0.5 - a$, $c \simeq e^{i2\pi/3}$ from numerical calculations. $P'$ only has six non-zero eigenvalues, therefore the ground states are six-fold degenerate. The corresponding eigenstates' $\pi/2$ rotation eigenvalues are $\pm 1$, $\pm 1$ and $\pm i$, consistent with the $SU(3)_2$ CS theory. Especially, by comparing the rotation eigenvalues of the modular $\mathcal{S}$-matrix with those of $P'$ eigenstates, one may obtain useful clues for the relations of the quasiparticle Wilson loop states with $|\Phi_x\Phi_y>$.